Wallpaper Groups

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1 Introduction

1.1 Scope

Our goal is to produce an expository paper proving that there exist exactly 17 Wallpaper Groups, built up from the knowledge from 4340. Due to space constraints we will not be able to prove this in full, instead we prove the important lemmas, theorems, and some representative cases.

1.2 Prerequisites

In addition to 4340, we assume knowledge of linear algebra at the level of 4330. In particular we will make use of the First Isomorphism Theorem for vector spaces and extensive use of orthogonal transformations.

2 Background

2.1 The Euclidean Group

Definition 1. An **isometry** is a distance preserving map from $\mathbb{R}^n$ to itself. Equivalently, it is an affine transformation

$$x \mapsto Mx + v$$

for some orthogonal $M \in \mathcal{M}_n$ and $v \in \mathbb{R}^n$. [5] [2]

We won't prove the equivalence of the definitions, see [2] for a proof of this fact.

Lemma 2. Any isometry g is invertible, and its inverse is also an isometry.

Proof. Let $T(x) = Mx + v$ be an isometry. Then let $H(x) = M^T(x - v)$. Then H is an inverse of T. Moreover $H(x) = M^T x - M^T v$, with M^T orthogonal and $M^T v \in \mathbb{R}^2$. □

Definition/Lemma 3. The **Euclidean Group**, E_2, is the group of isometries of the plane under composition

Proof.

- clearly the identity map, ι preserves distances so it is in E_2, moreover

$$(\iota \circ g)(x) = \iota(g(x)) = g(x) = g(\iota(x)) = (g \circ \iota)(x)$$

  for all $g \in E_2$

- By lemma 2, the inverse axiom is satisfied.
- The associative law holds since function composition is associative

□

Definition/Lemma 4. Let T be the subgroup of translations. Let O be the subgroup of orthogonal transformations (rotations about the origin and reflections across lines through the origin). Then $E_2 = T \circ O$

Proof. By definition, any $\phi \in E_2$ is of the form $\phi(x) = Mx + v$, in particular $\phi(x) = (\alpha \circ \beta)(x)$ where $\beta(x) = Mx$ and $\alpha(x) = x + v$.

The proof that T, O are subgroups is an exercise, it follows from basic facts from MATH 4330. $\square$

Lemma 5. T is a normal subgroup of E_2.

Proof. Take $f \in O$, $\tau \in T$ with $\tau(\mathbf{0}) = \mathbf{v}$.

As the orthogonal group O is linear, we have that f is linear.

Then,

$$\begin{aligned}(f \circ \tau \circ f^{-1})(x) &= f(f^{-1}(x) + v) \\ &= f(f^{-1}(x)) + f(v) \\ &= x + f(v)\end{aligned}$$

which is a translation. Translations also commute with other translations, so we know that $\tau' \tau \tau'^{-1} = \tau$ for any $\tau, \tau' \in T$. As T and O together generate E_2, we thus have that T is a normal subgroup of E_2. $\square$

Definition 6. Let $f(x) = Mx + v$ if $\det(M) = 1$, f is a **direct isometry**. Otherwise ($\det(M) = -1$ and), f is an opposite isometry.

Alternatively, let $g = \tau f$ where $\tau \in T, f \in O$. Then, if f is a rotation then g is a **direct isometry**, and if f is a reflection then g is an **opposite isometry**.

Definition 7. A **glide reflection** is a reflection over some line m followed by a translation parallel to that line.

Theorem 8. Every direct isometry is either a translation or a rotation. Every opposite isometry is a reflection or a glide reflection.

Proof. Any direct isometry is represented by the pair (v, A), where A is a rotation of angle $0 \leqslant \theta < 2\pi$. If $\theta = 0$, this is just a translation.

Otherwise, $0 < \theta < 2\pi$, so consider $I - A$. Using the formula for a rotation matrix

$$I - A = \begin{bmatrix} 1 - \cos\theta & \sin\theta \\ -\sin\theta & 1 - \cos\theta \end{bmatrix}$$

It has determinant $2 - 2\cos\theta > 0$. Thus $I - A$ is invertible so $c - Ac = (I - A)c = v$ has a unique solution c. Now, $(v, A) = (c - Ac, A)$ and

$$x \mapsto (Ax + c - Ac = A(x - c) + c)$$

is a rotation about c.

Any opposite isometry is represented by the pair (v, B), where B is a reflection over a line l through the origin. Assume that v is not parallel to this line since if so we are done (in particular $Bv \neq v$).

If $Bv = -v$ then B is a reflection in the line l perpendicular to v. Let $a = v/2$ then (v, B) is a reflection in the line $m = l + a$. Indeed

$$a + B(x - a) = a + Bx - Ba = a + Bx - B\frac{1}{2}v = 2a + Bx$$

If instead $B(v) \neq -v, v$ then let $w := v - Bv$. Then

$$Bw = B(v - Bv) = Bv - B^2v = Bv - v = -w$$

Then let a be half the projection of v onto w, i.e. $a := \frac{1}{2}\frac{v\cdot w}{\|w\|^2}w$. Then let $b = v - 2a$, so that b is orthogonal to w (and hence parallel to to the line l).

Then $(v, B) = (2a + b, B)$ and it is a reflection over the line $l + a$ followed by a translation by b (parallel to it). Indeed since a is a scaling of w then since B is linear, $Bw = -w \implies Ba = -a$ so

$$B(x - a) + a + b = Bx - Ba + a + b = Bx + 2a + b$$

□

3 The Wallpaper Groups

Definition/Lemma 9. Define the homomorphism $\pi : E_2 \to O_2$ by

$$\pi(x \mapsto Mx + v) = M$$

If G is a subgroup of E_2, we write H for $G \cap T$ and J for $\pi(G)$

we call H the ***translational subgroup*** of G and J the ***point group*** of G.

Proof. We write (v, M) for the map $x \mapsto Mx+v$. To verify that π is a homomorphism, we have that

$$\begin{aligned}\pi((v, M)(v_1, M_1)) &= \pi(x \mapsto v + M(v_1 + M_1 x)) \\ &= \pi(v + Mv_1, MM_1) \\ &= MM_1 \\ &= \pi(v, M)\pi(v_1, M_1)\end{aligned}$$

□

Definition 10. A subgroup of E_2 is a **wallpaper group** if its translational subgroup is generated by two independent translations and its point group is finite.

Lemma 11. All finite subgroups of O are either cyclic or dihedral.

Proof. Let G be some finite non-trivial subgroup of O_2.

1.
   - First, suppose that $G \subseteq SO_2$ such that every element of G is a rotation of the plane. Write R_θ for the matrix representing clockwise rotation by θ about the origin for $0 \leqslant \theta < 2\pi$.
   - Choose $R_\phi \in G$ with the smallest positive ϕ. We know that for any $R_\theta \in G$, we can use euclidean division to divide θ by ϕ such that $\theta = k\phi + \psi$ for $k \in \mathbb{Z}$ and $0 \leqslant \psi < \phi$.
   - We get that
     $$R_\theta = R_{k\phi+\psi} = R_\phi^k R_\psi$$
     and thus $R_\psi = R_\phi^{-k} R_\theta \in G$. As ϕ is defined to be the smallest possible, we have that $0 \leqslant \psi < \phi \implies \psi = 0$. Therefore, $R^\theta = R_\phi^k$ and G is cyclic.
2.
   - Let G not be entirely inside SO_2. Let $H = G \cap SO_2$ and $K = G \backslash H$.
   - Choose $k \in K \notin H$. k is some matrix of determinant -1.
   - Take the set of elements $S = \{kh \mid h \in H\} \subseteq K$. These are all matrices with determinant -1. We want to show that $K \subseteq S \implies S = K$.
   - Let $k' \in K$. We know that $k^{-1} \in K$ and therefore that $k^{-1}k'$ has determinant 1 and is in H. Then, we see that $k^{-1}k' = h$ for some $h \in H$ and thus $k' = kh \in S$. Therefore, $K = S$ and so $|K| = |S| = |H|$ and so $[G : H] = [G : K] = 2$.
   - Now, choose a generator A for H and some element B from $G - H$.
   - B represents a reflection and thus $B^2 = I$. Thus, the elements of G are
     $$I, A, \cdots, A^{n-1}, B, AB, \cdots, A^{n-1}B$$

and they satisfy $A^n = I$, $B^2 = I$, $BA = A^{-1}B$. Thus, $A \to r$, $B \to s$ is an isomorphism between G and the dihedral group D_n.

□

3.1 Characterizing the Wallpaper Groups

Let G denote a wallpaper group with translation subgroup H and point group J. Let L be the orbit of the origin under the action of H on $\mathbb{R}^2$ by (left) translation.

Theorem 12. Take $\mathbf{a}$ a non-zero vector of minimum length in L, then choose a linearly independent non-zero vector $\mathbf{b}$ in L whose length is as small as possible.

Then, set L is the ***lattice*** spanned by a and b. That is to say, L consists of all linear combinations $m\mathbf{a} + n\mathbf{b}$ where $m, b \in \mathbb{Z}$

Proof. See figure 1

- $m\mathbf{a} + n\mathbf{b} \subseteq L$

  The map $(v, I) \mapsto v$ is an isomorphism between T and the additive group $\mathbb{R}^2$ sending H to L. In other words, the action $T \rightharpoondown \mathbb{R}^2$ is defined by $(v, I) \mapsto v \overset{\text{left}}{\rightharpoondown} \mathbf{0}$ is an isomorphism, so we know that H is a subgroup of T and thus L is a subgroup of $\mathbb{R}^2$.

  Therefore, because $\mathbf{a}, \mathbf{b} \in L$, we know that $m\mathbf{a} + n\mathbf{b} \subseteq L$.

- $L \subseteq m\mathbf{a} + n\mathbf{b}$

  Partition the plane into parallelograms using the lattice spanned by $\mathbf{a}$ and $\mathbf{b}$, as shown in Figure 1.

  If $\mathbf{x} \subseteq L$ is not in the lattice, then we know that it lies in one of the parallelograms in the lattice. This means that its distance from its nearest neighbor $|\mathbf{c}|$ is less than $|\mathbf{b}|$. As shown in the diagram, this is because the entire parallelogram can be covered by two circles of radius $|\mathbf{b}|$ centered at opposite corners as $|\mathbf{b}| \geqslant |\mathbf{a}|$.

  Let $\mathbf{w} = \mathbf{x} - \mathbf{c}$. We can write $\mathbf{x} = \tau_1(\mathbf{0})$ and $\mathbf{c} = \tau_2(\mathbf{0})$, and thus $\mathbf{w} = \tau_1(\mathbf{0}) - \tau_2(\mathbf{0}) = (\tau_1 - \tau_2)(\mathbf{0}) \in L$.

  If $|\mathbf{w}| < |\mathbf{a}|$ this is a contradiction as $|a|$ was chosen to be the vector of minimum length in L. If $|\mathbf{a}| \leqslant |\mathbf{w}| < |\mathbf{b}|$, then $\mathbf{w}$ must be linearly independent to $\mathbf{a}$ in order to lie inside the parallelogram, which contradicts the construction of $\mathbf{b}$, as $|\mathbf{w}| \in L < |\mathbf{b}|$.

  Therefore, such a $\mathbf{x}$ cannot exist and so $L \subseteq m\mathbf{a} + n\mathbf{b}$

$\therefore L = m\mathbf{a} + n\mathbf{b}$. □

Theorem 13. The point group J acts on the lattice L by conjugation.

Proof. If $M \in J$ and $\mathbf{x} \in L$, we just need to show that $M\mathbf{x} \in L$ since the other properties (associativity and identity) follow directly from properties of matrix multiplication.

Suppose that $\pi(g) = M$ for some $g = (\mathbf{v}, M)$, and let $\tau = (\mathbf{x}, I)$. H is the kernel of the homomorphism $\pi : G \to J$, and thus it is a normal subgroup of G, and so $g\tau g^{-1} \in H$. Moreover, we have that

$$\begin{aligned} g\tau g^{-1} &= (\mathbf{v}, M)(\mathbf{x}, I)(\mathbf{v}, M)^{-1} \\ &= (\mathbf{v}, M)(\mathbf{x}, I)(-M^{-1}\mathbf{v}, M^{-1}) \\ &= (\mathbf{v}, M)(-M^{-1}\mathbf{v} + \mathbf{x}, M^{-1}) \\ &= (\mathbf{v} - MM^{-1}\mathbf{v} + M\mathbf{x}, I) \\ &= (M\mathbf{x}, I) \end{aligned}$$

This is a translation $\in H$, so Mx is a member of L. □

Theorem 14. The order of a nontrivial rotation in a wallpaper group can only be $2, 3, 4$, or 6

Proof. Let $B \in J$, the point group. Since the point group of any wallpaper group is finite B has finite order. So let q be the order of B. Then either B is clockwise or anticlockwise. If it is anticlockwise then B is a rotation of angle $2\pi/q$ if it is anticlockwise then B^{q-1} is a rotation of angle $2\pi/q$. So consider $A \in J$ where

$$A = \begin{bmatrix} \cos(\frac{2\pi}{q}) & -\sin(\frac{2\pi}{q}) \\ \sin(\frac{2\pi}{q}) & \cos(\frac{2\pi}{q}) \end{bmatrix}$$

Let a be a nonzero shortest vector in the lattice, L of the wallpaper group. Since J acts on L (Theorem 13), $Aa \in L$. Suppose for a contradiction that $q > 6$. Then $2\pi/q < 2\pi/6$, so $\cos(2\pi/q) > 1/2$. Now we claim that $Aa - a \in L$ is shorter than a, contradicting the definition of a.

First calculate

$$Aa = \begin{bmatrix} \cos(\frac{2\pi}{q}) & -\sin(\frac{2\pi}{q}) \\ \sin(\frac{2\pi}{q}) & \cos(\frac{2\pi}{q}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1\cos(\frac{2\pi}{q}) - a_2\sin(\frac{2\pi}{q}) \\ a_1\sin(\frac{2\pi}{q}) + a_2\cos(\frac{2\pi}{q}) \end{bmatrix}$$

then

$$\begin{aligned}
\|Aa - a\|^2 &= \|Aa\|^2 - 2\langle Aa, a\rangle + \|a\|^2 \\
&= 2\|a\|^2 - 2\langle Aa, a\rangle \\
&= 2\|a\|^2 - 2\left(a_1^2\cos(\frac{2\pi}{q}) - a_2a_1\sin(\frac{2\pi}{q}) + a_1a_2\sin(\frac{2\pi}{q}) + a_2^2\cos(\frac{2\pi}{q})\right) \\
&= 2\|a\|^2 - 2\left(a_1^2\cos(\frac{2\pi}{q}) + a_2^2\cos(\frac{2\pi}{q})\right) \\
&< 2\|a\|^2 - 2\left(\frac{1}{2}a_1^2 + a_2^2\frac{1}{2}\right) \\
&= \|a\|^2
\end{aligned}$$

Hence the order of any element must be at most 6. Suppose for a contradiction $q = 5$. Then by a similar calculation, $A^2a - a$ is shorter than a. □

Corollary 15 (of theorem 14 and lemma 11)**.** The point group of a wallpaper group is generated by a rotation through one of the angles $0, \pi, 2\pi/3, \pi/2, \pi/3$ and possibly a reflection.

Lemma 16. An isomorphism preserves order.

Proof. In particular, if g has order n then $(\phi(g))^n = \phi(g^n) = \phi(e) = e$ so order of $\phi(g)$ is finite and divides n in the other direction, let r be the order of $\phi(g)$, we have the inverse isomorphism, $g^r = (\phi^{-1}(\phi(g)))^r = \phi^{-1}(\phi(g)^r) = \phi^{-1}(e) = e$. So $n|r$. Since $n|r$ and $r|n$ then $r = n$. □

Theorem 17. An isomorphism between wallpaper groups takes translations to translations, rotations to rotations, reflections to reflections and glide reflections to glide reflections

Proof. Let $\phi : G \to G_1$ be an isomorphism between wallpaper groups, and let τ be a translation in G.

Translations and glides have infinite order, whereas rotations and reflections have finite order, thus $\phi(\tau)$ is either a translation or a glide by theorem 16.

If $\phi(\tau)$ is a glide then let τ_1 be a translation whose direction is not parallel to the glide, then τ_1 does not commute with $\phi(\tau)$, for the same reason, τ_1^2 does not commute with $\phi(\tau)$. Since ϕ is an isomorphism, the there is an element $g \in G$ such that $\phi(g) = \tau_1$. Again, g must be either a translation or a glide. Then g^2 is a translation hence it commutes with τ, but then

$$\tau_1^2\phi(\tau) = \phi(g^2)\phi(\tau) = \phi(g^2\tau) = \phi(\tau g^2) = \phi(\tau)\phi(g^2) = \phi(\tau)\tau_1^2$$

which is a contradiction. We have shown ϕ takes translations to translations and glides to glides.

Reflections have order 2, hence the image of a reflection under isomorphism is either a reflection or a half turn by lemma 16. Let $g \in G$ be a reflection s.t. $\phi(g)$ is a half turn, and choose $\tau \in G$ in a direction which is not perpendicular to the mirror of g. Then τg is a slide by proof of theorem 8. But $\phi(\tau g) = \phi(\tau)\phi(g)$ is the product of a translation and a half-turn which is another half-turn. This is a contradiction, so reflections correspond to reflections. Finally, rotations are forced to correspond to rotations. □

Corollary 18. If two wallpaper groups are isomorphic, then their point groups are also isomorphic.

Proof. Let G, G_1 be wallpaper groups with translation subgroups H, H_1 and point groups J, J_1 respectively. If $\phi : G \to G_1$ is an isomorphism then $\phi(H) = H_1$ by theorem 17. So ϕ induces an isomorphism from G/H to G_1/H_1. Then $J \simeq G/H \simeq G_1/H_1 \simeq J_1$. □

4 Wallpaper patterns

There are seventeen different wallpaper groups. We do not have the space to go through all of them, so instead we will focus on a few examples.

We classify the lattice into 5 types based on the shape of the basic parallelogram determined by **a** and **b**.

Definition 19. Lattice Type, pick **a**, **b** as in theorem 12, wlog replace **b** with **-b** so that

$$\|a - b\| \leqslant \|a + b\|$$

The 5 lattice types are defined as follows (see figure 6):

Oblique:	$\|a\| < \|b\| < \|a-b\| < \|a+b\|$
Rectangular:	$\|a\| < \|b\| < \|a-b\| = \|a+b\|$
Centred Rectangular:	$\|a\| < \|b\| = \|a-b\| < \|a+b\|$
Square:	$\|a\| = \|b\| < \|a-b\| = \|a+b\|$
Hexagonal:	$\|a\| = \|b\| = \|a-b\| < \|a+b\|$

While it appears that we are missing the case where $\|a\| = \|b\| < \|a - b\| < \|a + b\|$, it turns out that the basic parallelogram is a rhombus,and since the diagonals bisect eachother at right angles, this is a centred rectangular structure based on vectors $a-b, a+b$. See figure 6

Now we introduce the internationally recognized naming of the wallpaper groups. Each name consists of letters from p,c,m,g and numbers 1,2,3,4,6. p refers to the lattice and is short for *primitive*, it refers to viewing the lattice as copies of the basic parallelogram with no lattice points inside. c is short for *centred lattice* and refers to taking a nonprimitive cell together with its center point as the basic building block. m is for *mirror* and g is for *glide reflection*. 1,2,3,4,6 indicate rotations of the corresponding order (order 1 is the identity).

Continuing notation from earlier, G is a wallpaper group, H the translational subgroup, J the point group, L the lattice, a, b vectors which span L as in Theorem 12. Wlog we may assume a lies on the x axis and b on the y, but now it may be that a, b swap names (b could be shorter than a).

We proceed by considering each type of lattice in turn.

Oblique The only orthogonal transformations which preserve L are identity and rotation by π about the origin, so the point group of G is a subgroup of $\{\pm I\}$. There are two subgroups: $\{I\}$ and $\{\pm I\}$.

(p1) If J only contains the identity matrix, G is simply generated by two independent translations so its elements have the form $(ma+nb, I)$ for $m, n \in \mathbb{Z}$.

(p2) If $J = \{\pm I\}$, then G contains a half turn (a rotation of angle π). Wlog let the origin be the fixed point of this rotation, so that $(0, -I) \in G$.

We claim that the union of the cosets $H \cup H(0, -I)$ is equal to G. Indeed, it must contain G since if $(v, M) \in G$, then M is $\pm I$. If $M = I$ then $(v, M) = (v, I) = v \in H$ and if $M = -I$ then since G is a group $(v, -I)(0, -I) = (v, I) \in G$ so $v \in H$ so $(v, -I) \in H(0, -I)$. On the other hand, it is a subset of G since by definition $H \subset G$ and since G is a group, and $(0, -I) \in G$ then $H(0, -I) \subset G$.

So any elements of G which are not translations are in $H(0, -I)$ and have the form

$$(ma + nb, I)(0, -I) = (ma + nb, -I)$$

for $m, n \in \mathbb{Z}$. I.e. in addition to the transformations in p1, we have all the half turns about the points $\frac{1}{2}ma + \frac{1}{2}nb$ (see proof of theorem 8, in this case we have $2c = ma + nb$ so the center of rotation is $1/2(ma + nb)$).

Rectangular There are 4 orthogonal transformations which preserve L: identity, half turn about 0, reflection in x-axis B_0, and reflection in y-axis B_π. Denote by B_ϕ a reflection in the line through the origin with an angle of $\phi/2$ with the positive x-axis. Ignoring those wallpaper posibilities we have previously seen, we have (pm) $J = \{I, B_0\}$, and G contains a reflection in a horizontal mirror (G contains $(0, B_0)$). As before, We claim that the union of the cosets $H \cup H(0, B_0)$ is equal to G. Indeed, it must contain G since if $(v, M) \in G$, then M is I, B_0. If $M = I$ then $(v, M) = (v, I) = v \in H$ and if $M = B_0$ then since G is a group $(v, B_0)(0, B_0) = (v, I) \in G$ so $v \in H$ so $(v, B_0) \in H(0, B_0)$. On the other hand, it is a subset of G since by definition $H \subset G$ and since G is a group, and $(0, B_0) \in G$ then $H(0, B_0) \subset G$.

(pg) $J = \{I, B_0\}$, but this time, there are no reflections in G ($(0, B_0) \notin G$). So G must contain a glide reflection whose line is horizontal, and we choose a point of this line as the origin. A glide reflection composed with itself is a translation, so the glide has the form $(\frac{1}{2}ka, B_0)$ for some integer k. If k is even then

$$(0, B_0) = (-\frac{1}{2}ka, I)(\frac{1}{2}ka, B_0) \in G$$

A contradiction, hence k is odd and

$$(\frac{1}{2}a, B_0) = (-\frac{1}{2}(k-1)a, I)(\frac{1}{2}ka, B_0) \in G$$

As before elements of G are either in H or $H(\frac{1}{2}a, B_0)$. Suppose $(v, M) \in G$, if $M = I$ we are done, so assume $M = B_0$. Then

$$(v, B_0)(-\frac{1}{2}a, B_0) = (-\frac{1}{2}B_0 a + v, I) = (-\frac{1}{2}a + v, I)$$

Next, since $(-\frac{1}{2}a + v, I) \in H$, $(-\frac{1}{2}a + v, I)(\frac{1}{2}a, B_0) = (v, B_0) \in H(\frac{1}{2}a, B_0)$. The other inclusion is trivial, see the same argument for (pm) above.

Note that interchanging horizontal and vertical with B_π instead of B_0 gives an isomorphic group.

While there are yet 17-4 = 13 more wallpaper groups to describe, and we unfortunately don't have room to fit them all, so we will conclude by proving that the four wallpaper groups we have thus far observed are in fact all distinct.

Theorem 20. No two of p1, p2, pm, and pg are isomorphic.

Proof. By way of contradiction, since the point group of p1 is trivial while the point groups of the others are nontrivial, we obtain a contradiction by corollary 18. Since p2 is the only one which contains rotations it cannot be isomorphic to the others by theorem 17. Similarly, pg does not contain a reflection, while pm does. $\square$

Figures and illustrations

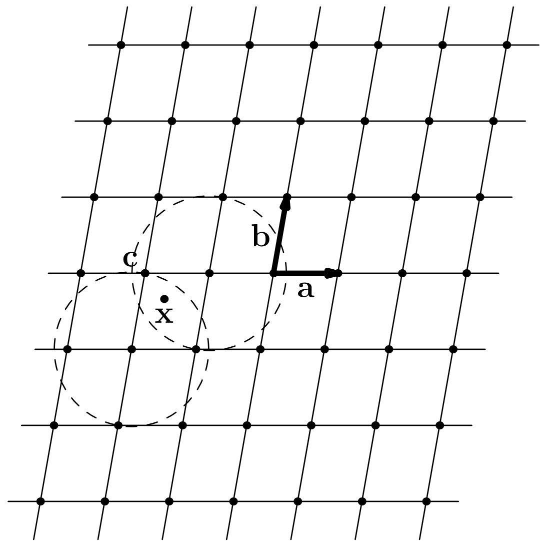

Figure 1: $\mathbf{x}$ on the lattice $m\mathbf{a} + n\mathbf{b}$

Figure 2: p1 wallpaper, *From The Grammar of Ornament (1856), by Owen Jones. "Middle Ages No 3" (plate 68), image #19.*

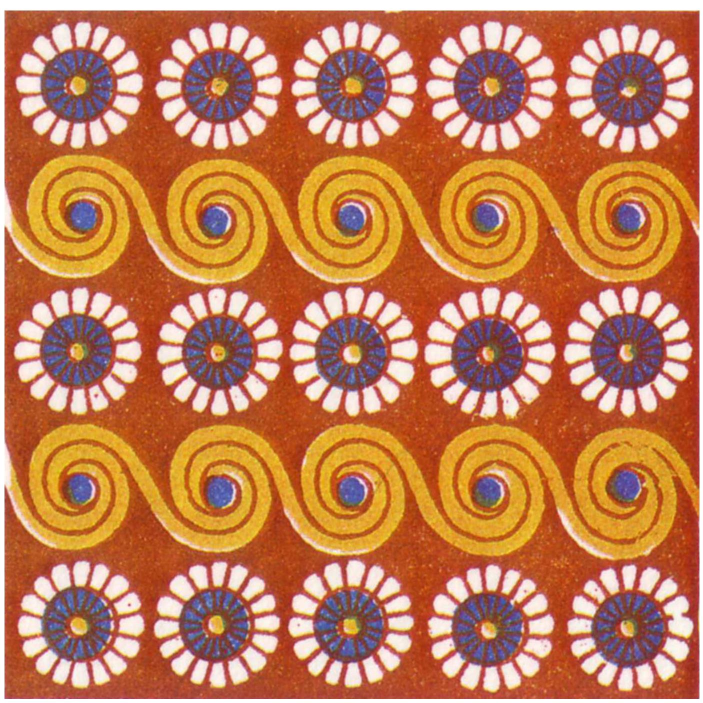

Figure 3: p2 wallpaper, *by Owen Jones*

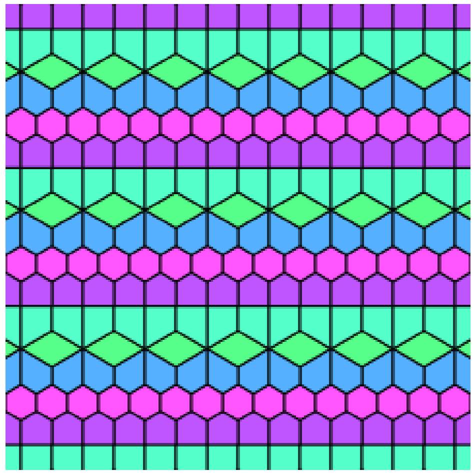

Figure 4: pm wallpaper, *by Harry Princeton*

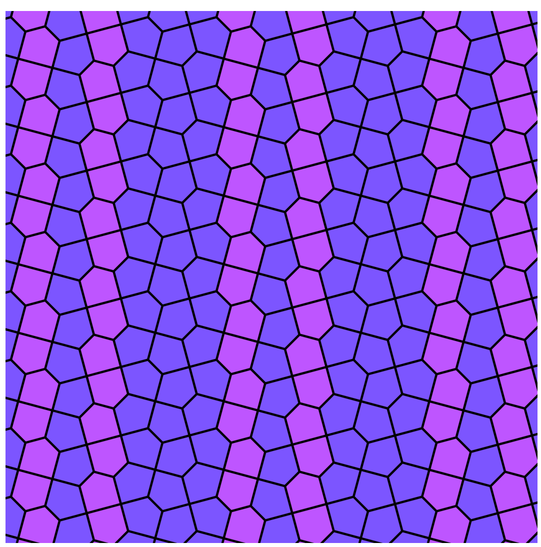

Figure 5: pg wallpaper, *by Harry Princeton*

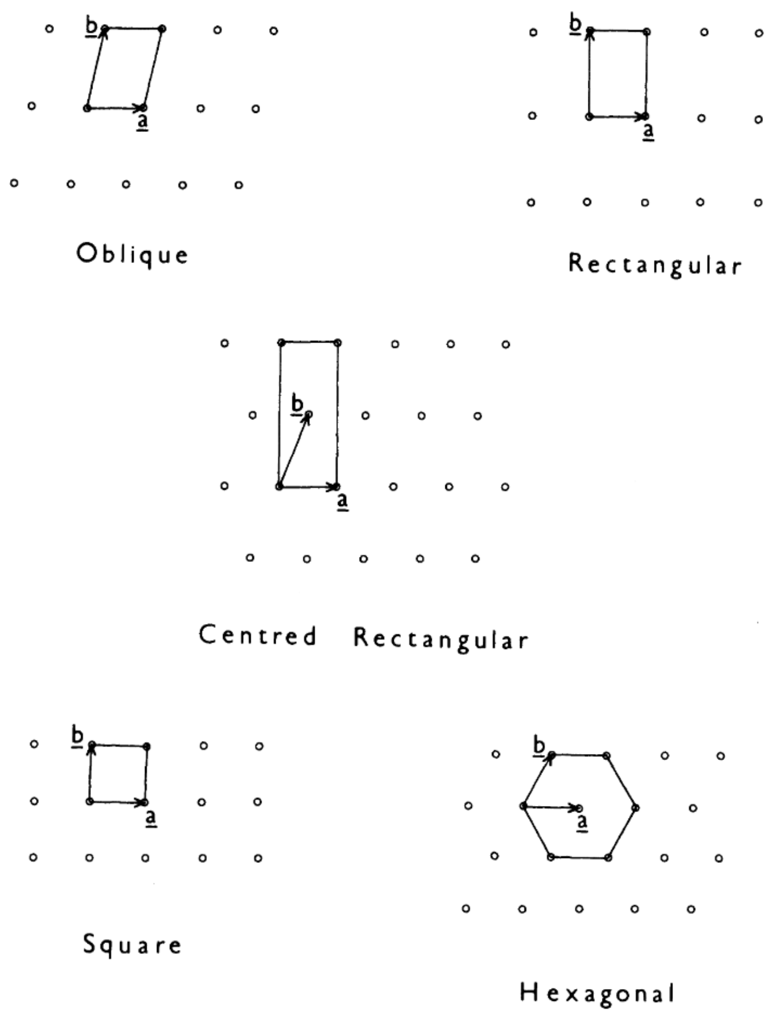

Figure 25.5

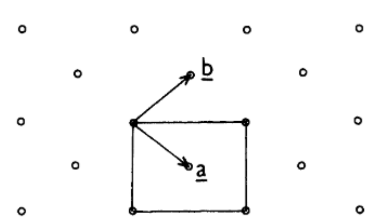

Figure 25.6

Figure 6: Lattice Types, illustration from Armstrong [1]

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