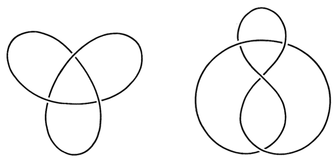
Presentation of Knot Group MATH 4530 Elijah Blum

Knots, preliminaries

Def. *A **knot** is an embedding of the circle S^1 into $\mathbb{R}^3$.*

Warning: *It is also common to refer to the image of the embedding as the knot itself*

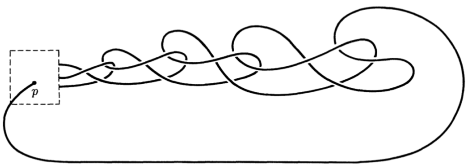


Two knots, the trefoil and the figure-eight knot

Def. *Two knots, K_1, K_2, are **equivalent** if there exists a homeomorphism of $\mathbb{R}^3$ onto itself which maps K_1 onto K_2*

Knottedness is not a topological property![1]

Warning: *We will only consider '**tame knots**', i.e. knots which are equivalent to **polygonal knots**, i.e. a knot whose image is the union of finitely many closed straight line segments. For example the following knot is wild (not tame)*



an example of a wild knot from [1]

A sufficient condition to be tame is that the embedding is C^1.

Theorem. ***(Gordon and Luecke)*** *Two knots are equivalent iff their complements are homeomorphic [2]*

Def. *The knot group, $G(K)$, of a knot K is the fundamental group of the complement of the knot, $\pi_1(\mathbb{R}^3 \backslash K)$*[2]*.*

Def. *Let $\mathcal{P}$ be a parallel projection of $\mathbb{R}^3$ onto a plane (ex $\mathcal{P}(x, y, z) := (x, y, 0)$), K a knot, then a point p has order cardinality of $\mathcal{P}^{-1}p \cap K$. A polygonal knot, K, is in regular position if (i) the only multiple points of K are double points, and there are only finitely many of them and (ii) no double point is in the image of a vertex of K.*

Theorem. *Any polygonal knot K is equivalent under an arbitrarily small rotation of $\mathbb{R}^3$ to a polygonal knot in regular position.[1]*

Writinger Presentation

Note the knot complement is path connected so we can pick any base point.

Given a knot in regular position projected onto the plane, it is intuitively clear that $\pi_1(\mathbb{R}^3/K)$ is generated by loops a_i which pass around the arcs.

Choose an orientation for the knot and orient a_i by right hand rule. Then consider the crossings, there are two types, each giving relations. An illustration from [7].

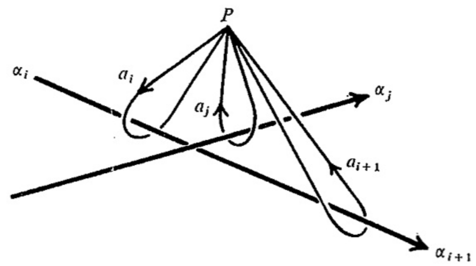


Figure 157

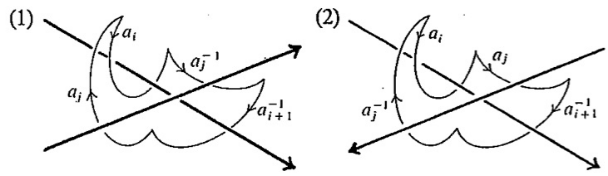


Figure 158

For type (1), notice that $a_i a_j^{-1} a_{i+1}^{-1} a_j$ is nulhomotopic. Thus, we have the relations:

$$a_i a_j^{-1} a_{i+1}^{-1} a_j = 1 \qquad a_i a_j a_{i+1}^{-1} a_j^{-1} = 1$$

for types (1) and (2) respectively. In fact these are all the relations, in particular,

Theorem. *$\langle$ arc labels | crossing relations $\rangle$ is a presentation of the knot group.*

One of these relations is redundant. Examples:

$$\pi(R^3 \backslash S^1) = \langle x \rangle = \mathbb{Z}$$
$$\pi(R^3 \backslash trefoil) = \langle x, y | yxy = xyx \rangle$$

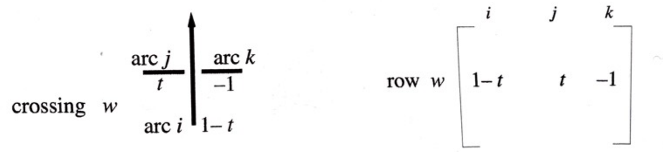
The full proof relies on the Seifert-Van Kampen Theorem

[1] of the knot as a subspace

[2] it is common to omit the base point since the complement of a knot is path-connected

Alexander Polynomial

Figure 6.1.1: The Alexander polynomial matrix entries for crossing w.



Def. *The **Alexander Matrix** of an oriented and labeled diagram is the $(n-1)\times(n-1)$ matrix with entries described by the above and the last row and column deleted. The **Alexander polnomial** is the determinant of the Alexander matrix.*

Theorem. *If D and D' are two different diagrams of an oriented knot K, then the Alexander polynomials of the two diagrams will differ by a factor of $\pm t^k$ for some $k \in \mathbb{Z}$.*

Proof Idea Changing labelings of the edges and crossings permutes the rows and colums of the Alexander matrix, and two diagrams of the same knot will only differ by Reidemeister moves[3].

Example: *An* Alexander polynomial of the unknot is 1. *An* Alexander polynomial of the trefoil is $t^2 - t + 1$

$$\begin{bmatrix} 1-t & t & -1 \\ -1 & 1-t & t \\ t & -1 & 1-t \end{bmatrix}$$

A direct construction of the Alexander Matrix from the Writinger presentation relies on the Fox Calculus. In this case the proof that the Alexander Polynomial is an invariant relies on the Tieze theorem instead of on Reidemeister moves.

Knot invariants and limitations

A knot invariant ϕ is a function from the set of knots such that if K, K' are equivalent, then $\phi(K) = \phi(K')$[5].

Knot invariants are studied in order to distinguish knots and understand fundamental properties (and how they relate to other math)

knot type of K

$\downarrow$

presentation type of $\pi_1(\mathbb{R}^3 \backslash K)$

$\downarrow$

(sequence of elementary ideals)

$\downarrow$

sequence of knot polnomials

The only "complete" invariant is the first i.e. the knot type (by definition).

Here is an example of two knots of different type which are knot ;) distinguished by presentation type of $\pi_1(\mathbb{R}^3 \backslash K)$ [1]

(4.8) *Granny knot and Square knot* (Figures 54 and 55).

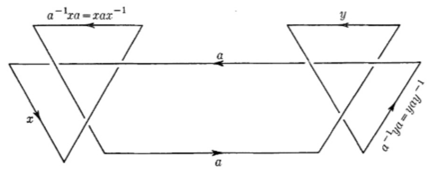


Figure 54

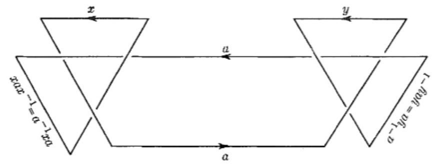


Figure 55

Here is an example of two knots of different type which are distinguished by presentation type of $\pi_1(\mathbb{R}^3 \backslash K)$ but knot ;) by their Alexander polynomials [1]

(4.7) (Figures 52 and 53).

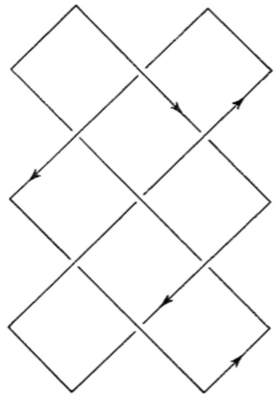


Figure 52

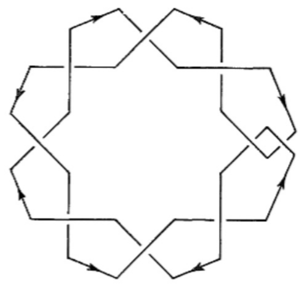


Figure 53

References

[1] R. H. Crowell, R. H. Fox, *Introduction to knot theory*, Springer, Graduate Texts 57, 1963.

[2] Cameron Gordon and John Luecke, *Knots are determined by their complements.* J. Amer. Math. Soc. 2 (1989), no. 2, 371–415.

[3] Knot theory. *Encyclopedia of Mathematics.* URL: http:\\encyclopediaofmath.org\index.php?title=Knot_theory&oldid=53739

[4] nLab authors. (2023). knot group. https:\\ncatlab.org\nlab\show\knot+group

[5] Purcell, J. S. (2020). Hyperbolic Knot Theory. https:\\arxiv.org\abs\2002.12652

[6] Inga Johnson, Allison Henrich, *An Interactive Introduction to Knot Theory*, Aurora: Dover Modern Math Originals, 2017.

[7] Stillwell, J. (1993). Knots and Braids. In: Classical Topology and Combinatorial Group Theory. Graduate Texts in Mathematics, vol 72. Springer, New York, NY. https:\\doi.org\10.1007\978-1-4612-4372-4_8

[3] Reidemeister (1927) proved that two knot diagrams of the same knot differ only by a sequence of three simple types of manipulations to the diagram called Reidemeister moves

BONUS: KNOT CHIRALITY

And a stricter definition of equivalence.

Def. *If f, g are embeddings in a space X, then an **ambient isotopy** taking f to g is a continuous map $F : X \times [0,1] \to X$ such that F_0 is the identity map, and $F_1 \circ f = g$*

*Two knots K_1, K_2 belong to the same **isotopy type** if there exists an ambient isotopy taking K_1 to K_2*

Two knots are equivalent (in the previous sense) if they are of the same isotopy type up to reflections.

Def. *A knot is **amphicheral** if it is equivalent to its mirror image*

The ***Laurent Polynomial*** can distinguish between *some* knots and their mirror image, while the Alexander Polynomial does not.[6]